SCATTERING THEORY FOR ENERGY-SUPERCRITICAL KLEIN-GORDON EQUATION

CHANGXING MIAO AND JIQIANG ZHENG

Abstract. In this paper, we consider the question of the global well-posedness and scattering for the cubic Klein-Gordon equation $u_{tt}-\Delta u+u+|u|^2u=0$ in dimension $d\geqslant 5$. We show that if the solution u is apriorily bounded in the critical Sobolev space, that is, $(u,u_t)\in L_t^\infty(I;H_x^{s_c}(\mathbb{R}^d)\times H_x^{s_c-1}(\mathbb{R}^d))$ with $s_c:=\frac{d}{2}-1>1$, then u is global and scatters. The impetus to consider this problem stems from a series of recent works for the energy-supercritical nonlinear wave equation and nonlinear Schrödinger equation. However, the scaling invariance is broken in the Klein-Gordon equation. We will utilize the concentration compactness ideas to show that the proof of the global well-posedness and scattering is reduced to disprove the existence of the scenario: soliton-like solutions. And such solutions are precluded by making use of the Morawetz inequality, finite speed of propagation and concentration of potential energy.

Key Words: Klein-Gordon equation; scattering theory; Strichartz estimate; Energy supercritical; concentration compactness
AMS Classification: 35P25, 35B40, 35Q40.

1. Introduction

This paper is devoted to the study of the Cauchy problem of the cubic Klein-Gordon equation

$$\begin{cases}\ddot{u}-\Delta u+u+f(u)=0, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\ d\geqslant 5,\\ \big(u(0,x),\ u_t(0,x)\big)=\big(u_0(x),\ u_1(x)\big)\in H^{s_c}(\mathbb{R}^d)\times H^{s_c-1}(\mathbb{R}^d),\end{cases}\tag{1.1}$$

where $f(u)=|u|^2u$, u is a real-valued function defined in $\mathbb{R}^{1+d}$, the dot denotes the time derivative, Δ is the Laplacian in $\mathbb{R}^d$, $s_c:=\frac{d}{2}-1$.

Formally, the solution u of (1.1) conserves the energy

$$\begin{aligned}E(u(t),\dot{u}(t))=&\frac{1}{2}\int_{\mathbb{R}^d}\Big(\big|\dot{u}(t,x)\big|^2+\big|\nabla u(t,x)\big|^2+\big|u(t,x)\big|^2\Big)dx+\frac{1}{4}\int_{\mathbb{R}^d}|u|^4dx\\ \equiv&E(u_0,u_1).\end{aligned}$$

The class of solutions to wave equation

$$\ddot{u}-\Delta u+|u|^2u=0$$

is left invariant by the scaling

$$u(t,x)\mapsto\lambda\ u(\lambda\ t,\lambda\ x),\quad\forall\ \lambda>0.\tag{1.2}$$

Moreover, it leaves the Sobolev norm $\dot{H}_x^{s_c}(\mathbb{R}^d)$ with $s_c=\frac{d}{2}-1$ invariant. Since $s_c>1$, it is called the energy-supercritical.

The scattering theory for the Klein-Gordon equation with $f(u)=\mu|u|^{p-1}u$ has been intensively studied in [3, 4, 12, 14, 26, 27]. For $\mu=1$ and

$$1+\frac{4}{d}<p<1+\gamma_d\frac{4}{d-2},\quad \gamma_d=\begin{cases}1, & 3\leqslant d\leqslant 9;\\ \dfrac{d}{d+1}, & d\geqslant 10.\end{cases}\tag{1.3}$$

Brenner [4] established the scattering results in the energy space $H^1_x(\mathbb{R}^d)\times L^2_x(\mathbb{R}^d)$, which does not contain all subcritical cases for $d\geqslant 10$. Thereafter, Ginibre and Velo [12] exploited the Birman-Solomjak space $\ell^m(L^q,I,B)$ in [2] and the delicate estimates to improve the results in [4], which covered all subcritical cases. Finally K. Nakanishi [26] obtained the scattering results for the critical case ($p=1+\frac{4}{d-2}$) by the strategy of induction on energy [9] and a new Morawetz-type estimate. And recently, S. Ibrahim, N. Masmoudi and K. Nakanishi [14, 15] utilized the concentration compactness ideas to give the scattering threshold for the focusing ($\mu=-1$) nonlinear Klein-Gordon equation. Their method also works for the defocusing case.

In this paper, we consider the cubic Klein-Gordon equation in dimension $d\geqslant 5$, which is the energy-supercritical case. Such results have been recently established for many other equations including the nonlinear wave equation (NLW) and the nonlinear Schrödinger equation (NLS), since Kenig-Merle [18] on NLW for radial solutions in $\mathbb{R}^3$. We also refer to [5–7, 10, 19–22].

To be more precise, let us recall the results for the energy-supercritical nonlinear wave equation

$$\ddot{u}-\Delta u+|u|^pu=0,\ (t,x)\in\mathbb{R}\times\mathbb{R}^d,\ p>\frac{4}{d-2}.$$

In dimension three, Kenig and Merle [18] proved that if the radial solution u is apriorily bounded in the critical Sobolev space, that is, $(u,u_t)\in L^\infty_t(I;\dot{H}^{s_c}_x(\mathbb{R}^d)\times\dot{H}^{s_c-1}_x(\mathbb{R}^d))$ with $s_c:=\frac{d}{2}-\frac{2}{p}>1$, then u is global and scatters. In [19], they also considered the radial solutions in odd dimensions. Later, Killip and Visan [21] showed the result in $\mathbb{R}^3$ for the non-radial solutions by making use of Huygens principal and so called "localized double Duhamel trick". Further, they [22] proved the radial solution in all dimensions in some ranges of p. Thereafter, Bulut [5–7] proved the results in dimensions $d\geqslant 5$ for the cubic nonlinearity (i.e. $p=2$). Recently, Duyckaerts, Kenig and Merle [10] obtain such result for the focusing wave equation with radial solution in three dimension. Their proof relies on the compactness/rigidity method, pointwise estimates on compact solutions obtained in [18], and channels of energy arguments used by the authors in previous works [11] on the energy-critical equation.

Before stating the main result, we introduce some background materials.

Definition 1.1 (solution). *A function $u:\ I\times\mathbb{R}^d\to\mathbb{R}$ on a nonempty time interval I containing zero is a strong solution to* (1.1) *if $(u,u_t)\in C^0_t(J;H^{s_c}_x(\mathbb{R}^d)\times H^{s_c-1}_x(\mathbb{R}^d))$ and $u\in[W](J)$* (*defined in* (1.8)) *for any compact interval $J\subset I$ and for each $t\in I$, it obeys the Duhamel formula:*

$$\begin{pmatrix}u(t)\\ \dot{u}(t)\end{pmatrix}=V_0(t)\begin{pmatrix}u_0(x)\\ u_1(x)\end{pmatrix}-\int_0^t V_0(t-s)\begin{pmatrix}0\\ f(u(s))\end{pmatrix}ds,\tag{1.4}$$

where

$$V_0(t) = \begin{pmatrix} \dot K(t), K(t) \\ \ddot K(t), \dot K(t) \end{pmatrix}, \quad K(t) = \frac{\sin(t\omega)}{\omega}, \quad \omega = \big(1-\Delta\big)^{1/2}.$$

We refer to the interval I as the lifespan of u. We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that u is a global solution if $I=\mathbb{R}$.

The solution lies in the space $[W](I)$ locally in time is natural since by Strichartz estimate, the linear flow always lies in this space. Also, if a solution u to (1.1) is global, with $\|u\|_{W(\mathbb{R})} < +\infty$, then it scatters in both time directions in the sense that there exist solutions $v_\pm$ of the free Klein-Gordon equation

$$\ddot v - \Delta v + v = 0 \tag{1.5}$$

with $(v_\pm(0), \dot v_\pm(0)) \in H_x^{s_c}(\mathbb{R}^d) \times H_x^{s_c-1}(\mathbb{R}^d)$ such that

$$\Big\| \big(u(t), \dot u(t)\big) - \big(v_\pm(t), \dot v_\pm(t)\big) \Big\|_{H_x^{s_c}\times H_x^{s_c-1}} \longrightarrow 0, \quad \text{as} \quad t \longrightarrow \pm\infty. \tag{1.6}$$

In view of this, we define

$$S_I(u) = \|u\|_{[W](I)} \tag{1.7}$$

as the scattering size of u, where

$$[W](I) = L_t^{\frac{2(d+1)}{d-1}}\big(I; B^{\frac{d-3}{2}}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)\big). \tag{1.8}$$

Closely associated with the notion of scattering is the notion of blowup:

Definition 1.2 (Blowup). *Let $u: I\times\mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to* (1.1). *If there exists a time $t_0 \in I$ such that $S_{[t_0, \sup I)}(u) = +\infty$, then we say that the solution u blows up forward in time. Similarly, if there exists a time $t_0\in I$ such that $S_{(inf\ I, t_0]}(u) = +\infty$, then we say that $u(t,x)$ blows up backward in time.*

Now we state our main result.

Theorem 1.1. *Assume that $d \geqslant 5$, and $s_c := \frac{d}{2} - 1$. Let $u:\ I\times\mathbb{R}^d \to \mathbb{R}$ be a maximal-lifespan solution to* (1.1) *such that*

$$\big\|(u, u_t)\big\|_{L_t^\infty(I; H_x^{s_c}(\mathbb{R}^d)\times H_x^{s_c-1}(\mathbb{R}^d))} < +\infty. \tag{1.9}$$

Then the solution u is global and scatters.

The outline of the proof of Theorem 1.1: For any $0 \leqslant E_0 < +\infty$, we define

$$L(E_0) := \sup\Big\{S_I(u):\ u:\ I\times\mathbb{R}^d \to \mathbb{R} \text{ such that } \sup_{t\in I}\big\|(u,u_t)\big\|^2_{H_x^{s_c}\times H_x^{s_c-1}} \leqslant E_0\Big\},$$

where the supremum is taken over all solutions $u:\ I\times\mathbb{R}^d \to \mathbb{R}$ to (1.1) satisfying $\big\|(u,u_t)\big\|^2_{H^{s_c}\times H^{s_c-1}} \leqslant E_0$. Thus, $L:\ [0,+\infty) \to [0,+\infty)$ is a non-decreasing function. Moreover, from the small data theory, see Theorem 2.1, we know that

$$L(E_0) \lesssim E_0^{\frac{1}{2}} \quad \text{for} \quad E_0 \leqslant \eta_0^2,$$

where $\eta_0 = \eta(d)$ is the threshold from the small data theory.

From the stability theory (see Theorem 2.5), we see that L is continuous. Therefore, there must exist a unique critical $E_c \in (0, +\infty]$ such that $L(E_0) < +\infty$ for $E_0 < E_c$ and $L(E_0) = +\infty$ for $E_0 \geqslant E_c$. In particular, if $u : I \times \mathbb{R}^d \to \mathbb{R}$ is a maximal-lifespan solution to (1.1) such that $\sup_{t\in I} \big\|(u, u_t)\big\|^2_{H^{s_c}\times H^{s_c-1}} < E_c$, then u is global and moreover,

$$S_{\mathbb{R}}(u) \leqslant L\big(\big\|(u, u_t)\big\|^2_{L_t^\infty(\mathbb{R};H^{s_c}\times H^{s_c-1})}\big).$$

The proof of Theorem 1.1 is equivalent to show $E_c = +\infty$. We argue by contradiction. We show that if $E_c < +\infty$, then there exists a nonlinear global solution of (1.1) with $L_t^\infty(\mathbb{R}; H_x^{s_c} \times H_x^{s_c-1})$-norm be exactly E_c. Moreover, this solution satisfies some strong compactness properties. This is completed in Section 4 where we utilize the profile decomposition that was established in Ibrahim, Masmoudi and Nakanishi [14], and a strategy introduced by Kenig and Merle [17]. Finally, we utilize the finiteness of the energy to show that the solutions obtained in Section 4 are not possible. More precisely, by Morawetz inequality [4, 23, 25]

$$\int_0^T \int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|} dxdt \lesssim E(u, u_t), \tag{1.10}$$

we know that the left-hand side of (1.10) is bounded by the energy for any $T > 0$. On the other hand, notice that by finite speed of propagation and concentration of potential energy, the left-hand side of (1.10) should grow logarithmical in time T. This gives a contradiction by choosing T sufficiently large.

The paper is organized as follows. In Section 2, we deal with the local theory for the equation (1.1). In Section 3, we give the linear and nonlinear profile decomposition and show some properties of the profile. Thereafter, we extract a critical solution in Section 4. Finally in Section 5, we preclude the critical solution, which completes the proof of Theorem 1.1.

We conclude the introduction by giving some notations which will be used throughout this paper. We always assume the spatial dimension $d \geqslant 5$ and $f(u) = |u|^2u$. For any $r, 1 \leqslant r \leqslant \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^d)$ and by r' the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$. For any $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^d)$ the usual Sobolev space. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that supp $\widehat{\psi} \subseteq \big\{\xi : \frac{1}{2} \leqslant |\xi| \leqslant 2\big\}$ and $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$. Define ψ_0 by $\widehat{\psi}_0 = 1 - \sum_{j\geqslant 1} \widehat{\psi}(2^{-j}\xi)$. Thus supp $\widehat{\psi}_0 \subseteq \big\{\xi : |\xi| \leqslant 2\big\}$ and $\widehat{\psi}_0 = 1$ for $|\xi| \leqslant 1$. We denote by Δ_j and $\mathcal{P}_0$ the convolution operators whose symbols are respectively given by $\widehat{\psi}(\xi/2^j)$ and $\widehat{\psi}_0(\xi)$. For $s \in \mathbb{R}, 1 \leqslant r \leqslant \infty$, the Besov spaces $B^s_{r,2}(\mathbb{R}^d)$ and $\dot{B}^s_{r,2}(\mathbb{R}^d)$ are defined by

$$\begin{aligned} B^s_{r,2}(\mathbb{R}^d) =& \Big\{u \in \mathcal{S}'(\mathbb{R}^d), \|\mathcal{P}_0 u\|^2_{L^r} + \big\|2^{js}\|\Delta_j u\|_{L^r}\big\|^2_{l^2_{j\in\mathbb{N}}} < \infty\Big\} \\ \dot{B}^s_{r,2}(\mathbb{R}^d) =& \Big\{u \in \mathcal{S}'_h(\mathbb{R}^d), \big\|2^{js}\|\Delta_j u\|_{L^r}\big\|^2_{l^2_{j\in\mathbb{Z}}} < \infty\Big\}. \end{aligned}$$

For details of Besov space, we refer to [1]. For any interval $I \subset \mathbb{R}$ and any Banach space X we denote by $\mathcal{C}(I; X)$ the space of strongly continuous functions from I to X and by $L^q(I; X)$ the space of strongly measurable functions from I to X with $\|u(\cdot); X\| \in L^q(I)$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2.

2. Preliminaries

2.1. **Strichartz estimate and local theory.** In this section, we consider the Cauchy problem for the equation (1.1)

$$\begin{cases} \ddot{u} - \Delta u + u + f(u) = 0, \\ u(0) = u_0, \ \dot{u}(0) = u_1. \end{cases} \tag{2.1}$$

The integral equation for the Cauchy problem (2.1) can be written as

$$u(t) = \dot{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds, \tag{2.2}$$

or

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t)\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s)\begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix}ds, \tag{2.3}$$

where

$$K(t) = \frac{\sin(t\omega)}{\omega}, \quad V_0(t) = \begin{pmatrix} \dot{K}(t), K(t) \\ \ddot{K}(t), \dot{K}(t) \end{pmatrix}, \quad \omega = \big(1-\Delta\big)^{1/2}.$$

Let $U(t) = e^{it\omega}$, then

$$\dot{K}(t) = \frac{U(t) + U(-t)}{2}, \qquad K(t) = \frac{U(t) - U(-t)}{2i\omega}.$$

Now we recall the following dispersive estimate for the operator $U(t) = e^{it\omega}$.

Lemma 2.1 ([4,12]). *Let $2 \leqslant r \leqslant \infty$ and $0 \leqslant \theta \leqslant 1$. Then*

$$\big\|e^{i\omega t}f\big\|_{B_{r,2}^{-(d+1+\theta)(\frac{1}{2}-\frac{1}{r})/2}} \leqslant \mu(t)\big\|f\big\|_{B_{r',2}^{(d+1+\theta)(\frac{1}{2}-\frac{1}{r})/2}},$$

where

$$\mu(t) = C\min\Big\{|t|^{-(d-1-\theta)(\frac{1}{2}-\frac{1}{r})_+}, |t|^{-(d-1+\theta)(\frac{1}{2}-\frac{1}{r})}\Big\}.$$

According to the above lemma, the abstract duality and interpolation argument(see [13], [16]), we have the following Strichartz estimates.

Lemma 2.2 ([4,12,23,24]). *Let $0 \leqslant \theta_i \leqslant 1$, $\rho_i \in \mathbb{R}$, $2 \leqslant q_i, r_i \leqslant +\infty$, $i = 1,2$. Assume that $(\theta_i, d, q_i, r_i) \neq (0,3,2,+\infty)$ satisfy the following admissible conditions*

$$\begin{cases} 0 \leqslant \dfrac{2}{q_i} \leqslant \min\Big\{(d-1+\theta_i)\Big(\dfrac{1}{2} - \dfrac{1}{r_i}\Big), 1\Big\}, \ i = 1,2 \\ \rho_1 + (d+\theta_1)\Big(\dfrac{1}{2} - \dfrac{1}{r_1}\Big) - \dfrac{1}{q_1} = \mu, \\ \rho_2 + (d+\theta_2)\Big(\dfrac{1}{2} - \dfrac{1}{r_2}\Big) - \dfrac{1}{q_2} = 1 - \mu. \end{cases} \tag{2.4}$$

Then, for $g \in H_x^\mu(\mathbb{R}^d)$, we have

$$\big\|U(\cdot)g\big\|_{L^{q_1}\big(\mathbb{R}; B_{r_1,2}^{\rho_1}\big)} \leqslant C\|g\|_{H^\mu}; \tag{2.5}$$

$$\big\|K_R * f\big\|_{L^{q_1}\big(I; B_{r_1,2}^{\rho_1}\big)} \leqslant C\big\|f\big\|_{L^{q_2'}\big(I; B_{r_2',2}^{-\rho_2}\big)}. \tag{2.6}$$

where the subscript R stands for retarded, and

$$K_R * f = \int_0^t K(t-s)f(u(s))ds.$$

Now it is useful to define several spaces and give estimates of the nonlinearities in terms of these spaces. Define

$$ST(I) = [W](I),$$

where

$$[W](I) = L_t^{\frac{2(d+1)}{d-1}}\big(I; B^{\frac{d-3}{2}}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)\big).$$

In addition to the ST-norm, we also need the corresponding dual norm

$$[W]^*(I) = L_t^{\frac{2(d+1)}{d+3}}\big(I; B^{\frac{d-3}{2}}_{\frac{2(d+1)}{d+3},2}(\mathbb{R}^d)\big).$$

Then we have by Strichartz estimate

$$\begin{aligned}&\big\|u\big\|_{[W](I)} + \big\|(u,u_t)\big\|_{L_t^\infty(I;H_x^{s_c}\times H_x^{s_c-1})}\\ \leqslant & C\big\|(u_0,u_1)\big\|_{H_x^{s_c}\times H_x^{s_c-1}} + C\big\|f(u)\big\|_{[W]^*(I)\oplus L_t^1(I;B^{s_c-1}_{2,1}(\mathbb{R}^d))},\end{aligned} \tag{2.7}$$

where the time interval I contains zero.

Lemma 2.3 (Product rule [8, 28])**.** *Let $s \geqslant 0$, and $1 < r, p_j, q_j < +\infty$ be such that $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ $(i = 1, 2)$. Then, we have*

$$\big\||\nabla|^s(fg)\big\|_{L_x^r(\mathbb{R}^d)} \lesssim \|f\|_{L_x^{p_1}(\mathbb{R}^d)}\big\||\nabla|^s g\big\|_{L_x^{q_1}(\mathbb{R}^d)} + \big\||\nabla|^s f\big\|_{L_x^{p_2}(\mathbb{R}^d)}\|g\|_{L_x^{q_2}(\mathbb{R}^d)}.$$

As a direct consequence, we have the following nonlinear estimate.

Lemma 2.4 (Nonlinear estimate)**.** *Let I be a time slab, one has*

$$\begin{aligned}&\big\|u^2v\big\|_{[W]^*(I)}\\ \lesssim&\big\|u\big\|^{1+\frac{2}{d-1}}_{[W](I)}\big\|u\big\|^{\frac{d-3}{d-1}}_{L_t^\infty\dot H_x^{s_c}}\big\|v\big\|^{\frac{2}{d-1}}_{[W](I)}\big\|v\big\|^{\frac{d-3}{d-1}}_{L_t^\infty\dot H_x^{s_c}} + \big\|v\big\|_{[W](I)}\big\|u\big\|^{\frac{4}{d-1}}_{[W](I)}\big\|u\big\|^{\frac{2(d-3)}{d-1}}_{L_t^\infty\dot H_x^{s_c}}.\end{aligned} \tag{2.8}$$

Proof. It follows from the above product rule and Sobolev embedding: $B^s_{p,\min\{p,2\}}(\mathbb{R}^d) \subset F^s_{p,2}(\mathbb{R}^d) = W_x^{s,p}(\mathbb{R}^d) \subset B^s_{p,\max\{p,2\}}(\mathbb{R}^d)$ that

$$\big\|u^2v\big\|_{[W]^*(I)} \lesssim \big\|u\big\|_{[W](I)}\big\|u\big\|_{L^{d+1}_{t,x}}\big\|v\big\|_{L^{d+1}_{t,x}} + \big\|v\big\|_{[W](I)}\big\|u\big\|^2_{L^{d+1}_{t,x}}. \tag{2.9}$$

Using Hölder's inequality and Sobolev embedding, we obtain

$$\begin{aligned}\big\|u\big\|_{L^{d+1}_{t,x}} \lesssim&\big\|u\big\|^{\frac{2}{d-1}}_{L_t^{\frac{2(d+1)}{d-1}}L_x^{\frac{2d(d+1)}{d+3}}}\big\|u\big\|^{\frac{d-3}{d-1}}_{L_t^\infty L_x^d}\\ \lesssim&\big\|u\big\|^{\frac{2}{d-1}}_{[W](I)}\big\|u\big\|^{\frac{d-3}{d-1}}_{L_t^\infty\dot H^{s_c}}.\end{aligned}$$

Plugging this into (2.9), we get (2.8). □

We can now state the local well-posedness for (1.1) with large initial data and small data scattering in the space $H^{s_c}(\mathbb{R}^d) \times H^{s_c-1}(\mathbb{R}^d)$, which is the first step to obtain the global time-space estimate and then lead to the scattering.

Theorem 2.1 (Local wellposedness). *Assume $(u_0, u_1) \in H_x^{s_c}(\mathbb{R}^d) \times H_x^{s_c-1}(\mathbb{R}^d)$. There exists a small constant $\delta = \delta(E)$ such that if $\|(u_0, u_1)\|_{H^{s_c}\times H^{s_c-1}} \leqslant E$ and I is an time interval containing zero such that*

$$\big\|\dot{K}(t)u_0 + K(t)u_1\big\|_{[W](I)} \leqslant \delta, \tag{2.10}$$

then there exists a unique strong solution u to (1.1) in $I\times\mathbb{R}^d$, with $u \in C(I; H_x^{s_c}(\mathbb{R}^d)) \cap C^1(I; H_x^{s_c-1}(\mathbb{R}^d))$ and

$$\|u\|_{[W](I)} \leqslant 2\delta, \ \ \|(u, u_t)\|_{L_t^\infty(I;H^{s_c}\times H^{s_c-1})} \leqslant 2CE, \tag{2.11}$$

where C is the Strichartz constant as in Lemma 2.2.

In particular, if $\|(u_0, u_1)\|_{H^{s_c}\times H^{s_c-1}} \leqslant \delta$, then the solution u is global and scatters.

Proof. We apply the Banach fixed point argument to prove this lemma. First we define the solution map

$$\Phi(u(t)) = \dot{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds \tag{2.12}$$

on the complete metric space B

$$B = \big\{u \in C(I; H^{s_c}) : \big\|(u, u_t)\big\|_{L_t^\infty(I;H^{s_c}\times H^{s_c-1})} \leqslant 2CE, \|u\|_{[W](I)} \leqslant 2\delta\big\}$$

with the metric $d(u, v) = \big\|u - v\big\|_{[W](I)\cap L_t^\infty H_x^{s_c}}$.

It suffices to prove that the operator defined by the RHS of (2.12) is a contraction map on B for I. If $u \in B$, then by Strichartz estimate (2.7), (2.8) and (2.10), we have

$$\begin{aligned}\big\|\Phi(u)\big\|_{[W](I)} \leqslant& \big\|\dot{K}(t)u_0 + K(t)u_1\big\|_{[W](I)} + C\big\|u^3\big\|_{[W]^*(I)}\\ \leqslant& \delta + C\big\|u\big\|_{[W](I)}^{1+\frac{4}{d-1}}\big\|u\big\|_{L_t^\infty \dot{H}^{s_c}}^{\frac{2(d-3)}{d-1}}.\end{aligned}$$

Plugging the assumption $\|u\|_{L^\infty(I;H^{s_c})} \leqslant 2CE$ and $\|u\|_{[W](I)} \leqslant 2\delta$, we see that for $u \in B$,

$$\big\|\Phi(u)\big\|_{[W](I)} \leqslant \delta + C(2\delta)^{1+\frac{4}{d-1}}(2CE)^{\frac{2(d-3)}{d-1}}.$$

Thus we can choose δ small depending on E and the Strichartz constant C such that

$$\big\|\Phi(u)\big\|_{[W](I)} \leqslant 2\delta.$$

Similarly, if $u \in B$, then $\big\|(\Phi(u), \partial_t\Phi(u))\big\|_{L_t^\infty(I;H^{s_c}\times H^{s_c-1})} \leqslant 2CE$. Hence $\Phi(u) \in B$ for $u \in B$. That is, the functional Φ maps the set B back to itself.

On the other hand, by a same argument as before and Lemma 2.4, we have for $u, v \in B$,

$$\begin{aligned}d(\Phi(u), \Phi(v)) \lesssim& C\|u^3 - v^3\|_{[W]^*(I)}\\ \leqslant& 16C\|u - v\|_{[W](I)\cap L_t^\infty H^{s_c}}\|(u, v)\|_{[W](I)}^{\frac{2}{d-1}}\big\|(u, v)\big\|_{[W](I)\cap L_t^\infty H^{s_c}}^{\frac{2(d-2)}{d-1}}\\ \leqslant& 16C(4\delta)^{\frac{2}{d-1}}(4CE + 2\delta)^{\frac{2(d-2)}{d-1}}d(u, v)\end{aligned}$$

which allows us to derive

$$d(\Phi(u),\Phi(v)) \leqslant \frac{1}{2}d(u,v),$$

by taking δ small such that

$$16C(4\delta)^{\frac{2}{d-1}}(4CE+2\delta)^{\frac{2(d-2)}{d-1}} \leqslant \frac{1}{2}.$$

A standard fixed point argument gives a unique solution u of (1.1) on $I\times\mathbb{R}^d$ which satisfies the bound (2.11). $\square$

Using Theorem 2.1 as well as its proof, one easily derives the following local theory for (1.1). We omit the standard detail here.

Theorem 2.2. *Assume that $d \geqslant 5$, $s_c = \frac{d}{2}-1$. Then, given $(u_0,u_1) \in H^{s_c}(\mathbb{R}^d)\times H^{s_c-1}(\mathbb{R}^d)$ and $t_0\in\mathbb{R}$, there exists a unique maximal-lifespan solution $u : I\times\mathbb{R}^d\to\mathbb{R}$ to (1.1) with initial data $\big(u(t_0),u_t(t_0)\big)=\big(u_0,u_1\big)$. This solution also has the following properties:*

1. *(Local existence) I is an open neighborhood of t_0.*
2. *(Blowup criterion) If $\sup(I)$ is finite, then u blows up forward in time (in the sense of Definition 1.2). If $\inf(I)$ is finite, then u blows up backward in time.*
3. *(Scattering) If $\sup(I)=+\infty$ and u does not blow up forward in time, then u scatters forward in time in the sense (1.6). Conversely, given $(v_+,\dot v_+)\in H^{s_c}(\mathbb{R}^d)\times H^{s_c-1}(\mathbb{R}^d)$ there is a unique solution to (1.1) in a neighborhood of infinity so that (1.6) holds.*

2.2. **Perturbation lemma.** In this part, we give the perturbation theory of the solution of (1.1) with the global space-time estimate.

With any real-valued function $u(t,x)$, we associate the complex-valued function $\vec{u}(t,x)$ by

$$\vec{u} = \langle\nabla\rangle^{s_c-1}\big(\langle\nabla\rangle u - i\dot{u}\big), \quad u = \Re\langle\nabla\rangle^{-s_c}\vec{u}, \tag{2.13}$$

where $\Re z$ denotes the real part of $z\in\mathbb{C}$. Then the free and nonlinear Klein-Gordon equations are given by

$$\begin{cases}(\Box+1)u=0 \Longleftrightarrow (i\partial_t+\langle\nabla\rangle)\vec{u}=0,\\ (\Box+1)u=-f(u) \Longleftrightarrow (i\partial_t+\langle\nabla\rangle)\vec{u}=-\langle\nabla\rangle^{s_c-1}f(\langle\nabla\rangle^{-s_c}\Re\vec{u}),\end{cases} \tag{2.14}$$

Lemma 2.5. *Let I be a time interval, $t_0\in I$ and $\vec{u},\vec{w}\in C(I;L^2(\mathbb{R}^d))$ satisfy*

$$(i\partial_t+\langle\nabla\rangle)\vec{u} = -\langle\nabla\rangle^{s_c-1}\big[f(u)+eq(u)\big]$$
$$(i\partial_t+\langle\nabla\rangle)\vec{w} = -\langle\nabla\rangle^{s_c-1}\big[f(w)+eq(w)\big]$$

for some function $eq(u), eq(w)$. Assume that for some constants $M,E>0$, we have

$$\big\|w\big\|_{ST(I)} \leqslant M, \tag{2.15}$$

$$\big\|\vec{u}\big\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} + \big\|\vec{w}\big\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \leqslant E, \tag{2.16}$$

Let $t_0\in I$, and let $(u(t_0),u_t(t_0))$ be close to $(w(t_0),w_t(t_0))$ in the sense that

$$\big\|\gamma_0\big\|_{ST(I)} \leqslant \epsilon, \tag{2.17}$$

where $\vec{\gamma}_0 = e^{i\langle\nabla\rangle(t-t_0)}(\vec{u}-\vec{w})(t_0)$ *and* $0<\epsilon<\epsilon_1=\epsilon_1(M,E)$ *is a small constant. Assume also that we have smallness conditions*

$$\big\|(eq(u), eq(w))\big\|_{ST^*(I)} \leqslant \epsilon, \tag{2.18}$$

where ϵ *is as above and*

$$ST^*(I) = [W]^*(I) \oplus L_t^1(I; B^{s_c-1}_{2,2}(\mathbb{R}^d)).$$

Then we conclude that

$$\begin{aligned} \big\|u-w\big\|_{ST(I)} \leqslant& C(M,E)\epsilon,\\ \big\|u\big\|_{ST(I)} \leqslant& C(M,E). \end{aligned} \tag{2.19}$$

Proof. Since $\|w\|_{ST(I)} \leqslant M$, there exists a partition of the right half of I at t_0:

$$t_0<t_1<\cdots<t_N,\ I_j=(t_j,t_{j+1}),\ I\cap(t_0,\infty)=(t_0,t_N),$$

such that $N\leqslant C(L,\delta)$ and for any $j=0,1,\cdots,N-1$, we have

$$\|w\|_{ST(I_j)} \leqslant \delta \ll 1. \tag{2.20}$$

The estimate on the left half of I at t_0 is analogue, we omit it.

Let

$$\gamma(t)=u(t)-w(t),\ \vec{\gamma}_j(t)=e^{i\langle\nabla\rangle(t-t_j)}\vec{\gamma}(t_j),\ 0\leqslant j\leqslant N-1, \tag{2.21}$$

then $\vec{\gamma}$ satisfies the following difference equation

$$\begin{cases} (i\partial_t+\langle\nabla\rangle)\vec{\gamma} = -\langle\nabla\rangle^{s_c-1}\Big(\big[\gamma(\gamma^2+3\gamma\omega+3\omega^2)\big]+eq(u)-eq(w)\Big)\\ \vec{\gamma}(t_j)=\vec{\gamma}_j(t_j), \end{cases}$$

which implies that

$$\begin{aligned} \vec{\gamma}(t) =& \vec{\gamma}_j(t) + i\int_{t_j}^{t} e^{i\langle\nabla\rangle(t-s)}\Big\{\langle\nabla\rangle^{s_c-1}\Big(\big[\gamma(\gamma^2+3\gamma\omega+3\omega^2)\big]+eq(u)-eq(w)\Big)\Big\}ds,\\ \vec{\gamma}_{j+1}(t) =& \vec{\gamma}_j(t) + i\int_{t_j}^{t_{j+1}} e^{i\langle\nabla\rangle(t-s)}\Big\{\langle\nabla\rangle^{s_c-1}\Big(\big[\gamma(\gamma^2+3\gamma\omega+3\omega^2)\big]+eq(u)-eq(w)\Big)\Big\}ds. \end{aligned}$$

By Strichartz estimate (2.7) and nonlinear estimate (2.8), we have

$$\begin{aligned} &\|\gamma-\gamma_j\|_{ST(I_j)} + \|\gamma_{j+1}-\gamma_j\|_{ST(\mathbb{R})}\\ \lesssim&\big\|\gamma^3+3\gamma^2\omega+3\gamma\omega^2\big\|_{[W]^*(I_j)} + \big\|(eq(u),eq(w))\big\|_{ST^*(I_j)}\\ \lesssim&\|\gamma\|_{ST(I_j)}^{1+\frac{4}{d-1}}\|\gamma\|_{L_t^\infty\dot H^{s_c}}^{\frac{2(d-3)}{d-1}} + \|\omega\|_{ST(I_j)}\|\gamma\|_{ST(I_j)}^{\frac{4}{d-1}}\|\gamma\|_{L_t^\infty\dot H^{s_c}}^{\frac{2(d-3)}{d-1}}\\ &+\|\gamma\|_{ST(I_j)}\|\omega\|_{ST(I_j)}^{\frac{4}{d-1}}\|\omega\|_{L_t^\infty\dot H^{s_c}}^{\frac{2(d-3)}{d-1}} + \big\|(eq(u),eq(w))\big\|_{ST^*(I_j)}. \end{aligned} \tag{2.22}$$

Therefore, assuming that

$$\|\gamma\|_{ST(I_j)} \leqslant \delta \ll 1,\ \forall\ j=0,1,\cdots,N-1, \tag{2.23}$$

then by (2.20) and (2.22), we have

$$\|\gamma\|_{ST(I_j)} + \|\gamma_{j+1}\|_{ST(t_{j+1},t_N)} \leqslant C\|\gamma_j\|_{ST(t_j,t_N)}+\epsilon, \tag{2.24}$$

for some absolute constant $C > 0$. By (2.17) and iteration on j, we obtain

$$\|\gamma\|_{ST(I)} \leqslant (2C)^N \epsilon \leqslant \frac{\delta}{2}, \tag{2.25}$$

provided we choose ϵ_1 sufficiently small. Hence the assumption (2.23) is justified by continuity in t and induction on j. Then repeating the estimate (2.22) once again, we can get the ST-norm estimate on γ, which implies the Strichartz estimates on u. $\square$

3. Profile decomposition

In this section, we first recall the linear profile decomposition of the sequence of L^2_x-bounded solutions of $(i\partial_t + \langle\nabla\rangle)\vec{v} = 0$ which was established in [14]. And then we show the nonlinear profile decomposition which will be used to construct the critical element and obtain its compactness properties in the next section.

3.1. **Linear profile decomposition.** First, we give some notation. For any triple $(t^j_n, x^j_n, h^j_n) \in \mathbb{R} \times \mathbb{R}^d \times (0, 1]$ with arbitrary suffix n and j, let τ^j_n, T^j_n, and $\langle\nabla\rangle^j_n$ respectively denote the scaled time shift, the unitary and the self-adjoint operators in $L^2(\mathbb{R}^d)$, defined by

$$\tau^j_n = -\frac{t^j_n}{h^j_n},\ T^j_n\varphi(x) = (h^j_n)^{-\frac{d}{2}}\varphi\Big(\frac{x - x^j_n}{h^j_n}\Big),\ \langle\nabla\rangle^j_n = \sqrt{-\Delta + (h^j_n)^2}. \tag{3.1}$$

Now we can state the linear profile decomposition as follows

Lemma 3.1 (Linear profile decomposition, [14])**.** *Let $\vec{v}_n(t) = e^{i\langle\nabla\rangle t}\vec{v}_n(0)$ be a sequence of free Klein-Gordon solutions with uniformly bounded $L^2_x(\mathbb{R}^d)$-norm. Then after replacing it with some subsequence, there exist $K \in \{0, 1, 2 \ldots, \infty\}$ and, for each integer $j \in [0, K)$, $\varphi^j \in L^2(\mathbb{R}^d)$ and $\{(t^j_n, x^j_n, h^j_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}\times\mathbb{R}^d\times(0, 1]$ satisfying the following. Define $\vec{v}^j_n$ and $\vec{\omega}^k_n$ for each $j < k \leqslant K$ by*

$$\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}^j_n(t, x) + \vec{\omega}^k_n(t, x), \tag{3.2}$$

where

$$\vec{v}^j_n(t, x) = e^{i\langle\nabla\rangle(t-t^j_n)}T^j_n\varphi^j(x) = T^j_n\Big(e^{i\langle\nabla\rangle^j_n\frac{t-t^j_n}{h^j_n}}\varphi^j\Big), \tag{3.3}$$

then we have

$$\lim_{k\to K}\overline{\lim_{n\to\infty}}\big\|\vec{\omega}^k_n\big\|_{L^\infty_t(\mathbb{R};B^{-\frac{d}{2}}_{\infty,\infty}(\mathbb{R}^d))} = 0, \tag{3.4}$$

and for any $l < j < k \leqslant K$ and any $t \in \mathbb{R}$,

$$\lim_{n\to\infty}\big\langle\mu\vec{v}^l_n, \mu\vec{v}^j_n\big\rangle^2_{L^2_x} = 0 = \lim_{n\to\infty}\big\langle\mu\vec{v}^j_n, \mu\vec{\omega}^k_n\big\rangle^2_{L^2_x}, \tag{3.5}$$

$$\lim_{n\to\infty}\Big\{\Big|\frac{h^l_n}{h^j_n}\Big| + \Big|\frac{h^j_n}{h^l_n}\Big| + \frac{|t^j_n - t^k_n| + |x^j_n - x^k_n|}{h^l_n}\Big\} = +\infty, \tag{3.6}$$

where $\mu \in \mathcal{MC}$ and $\mathcal{MC}$ is defined to be

$$\mathcal{MC} = \Big\{\mu = \mathcal{F}^{-1}\tilde{\mu}\mathcal{F}|\ \tilde{\mu} \in C(\mathbb{R}^d), \exists \lim_{|x|\to\infty}\tilde{\mu}(x) \in \mathbb{R}\Big\}.$$

Moreover, each sequence $\{h_n^j\}_{n\in\mathbb{N}}$ *is either going to* 0 *or identically* 1 *for all* n.

Remark 3.1. *We call* $\{\vec{v}_n^j\}_{n\in\mathbb{N}}$ *a free concentrating wave for each* j, *and* $\vec{w}_n^k$ *the remainder. From* (3.5), *we have the following asymptotic orthogonality*

$$\lim_{k\to K}\lim_{n\to+\infty}\Big(\|\mu\vec{v}_n(t)\|_{L^2}^2-\sum_{j=0}^{k-1}\|\mu\vec{v}_n^j(t)\|_{L^2}^2-\|\mu\vec{\omega}_n^k(t)\|_{L^2}^2\Big)=0. \tag{3.7}$$

We remark the following estimates for $1<p<\infty$,

$$\Big\|\big[|\nabla|-\langle\nabla\rangle_n\big]\varphi\Big\|_p \lesssim h_n\big\|\langle\nabla/h_n\rangle^{-1}\varphi\big\|_p, \tag{3.8}$$

hold uniformly for $0<h_n\leqslant 1$, by Mihlin's theorem on Fourier multipliers.

3.2. **Nonlinear profile decomposition.** After the linear profile decomposition of a sequence of initial data in the last subsection, we now show the nonlinear profile decomposition of a sequence of the solutions of (1.1) with the same initial data in the space $H^{s_c}(\mathbb{R}^d)\times H^{s_c-1}(\mathbb{R}^d)$.

First we construct a nonlinear profile associated with a free concentrating wave. Let $\vec{v}_n^j$ be a free concentrating wave for a sequence $(t_n^j,x_n^j,h_n^j)\in\mathbb{R}\times\mathbb{R}^d\times(0,1]$,

$$\begin{cases}(i\partial_t+\langle\nabla\rangle)\vec{v}_n^j=0,\\ \vec{v}_n^j(t_n)=T_n\varphi^j(x),\ \ \varphi^j(x)\in L^2(\mathbb{R}^d).\end{cases} \tag{3.9}$$

Then by Lemma 3.1, for a sequence of free Klein-Gordon solutions $\{\vec{v}_n(t)=e^{it\langle\nabla\rangle}\vec{v}_n(0)\}$ with uniformly bounded $L_x^2(\mathbb{R}^d)$-norm, we have a sequence of the free concentrating wave $\vec{v}_n^j(t,x)$ with $\vec{v}_n^j(t_n^j)=T_n^j\varphi^j$, $\varphi^j\in L^2(\mathbb{R}^d)$ for $j=0,1,\cdots,k-1$, such that

$$\begin{aligned}\vec{v}_n(t,x)=&\sum_{j=0}^{k-1}\vec{v}_n^j(t,x)+\vec{\omega}_n^k(t,x)\\ =&\sum_{j=0}^{k-1}e^{i\langle\nabla\rangle(t-t_n^j)}T_n^j\varphi^j(x)+\vec{\omega}_n^k(t,x)\\ =&\sum_{j=0}^{k-1}T_n^je^{i\left(\frac{t-t_n^j}{h_n^j}\right)\langle\nabla\rangle_n^j}\varphi^j+\vec{\omega}_n^k(t,x).\end{aligned}$$

Now for any free concentrating wave $\vec{v}_n^j$, we undo the group action T_n^j to look for the linear profile $\vec{V}_n^j$. Let

$$\vec{v}_n^j(t,x)=T_n^j\vec{V}_n^j\big((t-t_n^j)/h_n^j\big),$$

then we have

$$\vec{V}_n^j(t,x)=e^{it\langle\nabla\rangle_n^j}\varphi^j.$$

Next let $\vec{u}_n^j$ be the nonlinear solution with the same initial data $\vec{v}_n^j(0)$

$$\begin{cases}\big(i\partial_t+\langle\nabla\rangle\big)\vec{u}_n^j=-\langle\nabla\rangle^{s_c-1}f\big(\Re\langle\nabla\rangle^{-s_c}\vec{u}_n^j\big),\\ \vec{u}_n^j(0)=\vec{v}_n^j(0)=T_n^j\vec{V}_n^j(\tau_n^j),\end{cases} \tag{3.10}$$

where $\tau_n^j = -t_n^j/h_n^j$. In order to look for the nonlinear profile $\vec{U}_\infty^j$ associated with the free concentrating wave $\vec{v}_n^j$, we also need undo the group action. Define

$$\vec{u}_n^j(t,x) = T_n^j \vec{U}_n^j\big((t-t_n^j)/h_n^j\big),$$

then $\vec{U}_n^j$ satisfies the rescaled equation

$$\begin{cases} \big(i\partial_t + \langle\nabla\rangle_n^j\big)\vec{U}_n^j = -\big(\langle\nabla\rangle_n^j\big)^{s_c-1} f\big(\Re(\langle\nabla\rangle_n^j)^{-s_c}\vec{U}_n^j\big), \\ \vec{U}_n^j(\tau_n^j) = \vec{V}_n^j(\tau_n^j). \end{cases}$$

Extracting a subsequence, we may assume that there exist $h_\infty^j \in \{0,1\}$ and $\tau_\infty^j \in [-\infty,+\infty]$ for every j, such that as $n \to +\infty$

$$h_n^j \to h_\infty^j, \text{ and } \tau_n^j \to \tau_\infty^j.$$

Thus we have the limit equations as follows

$$\vec{V}_\infty^j = e^{it\langle\nabla\rangle_\infty^j}\varphi^j, \quad \begin{cases} \big(i\partial_t + \langle\nabla\rangle_\infty^j\big)\vec{U}_\infty^j = -\big(\langle\nabla\rangle_\infty^j\big)^{s_c-1} f\big(\hat{U}_\infty^j\big), \\ \vec{U}_\infty^j(\tau_\infty^j) = \vec{V}_\infty^j(\tau_\infty^j), \end{cases}$$

where $\hat{U}_\infty^j$ is denoted to be

$$\hat{U}_\infty^j := \Re\big(\langle\nabla\rangle_\infty^j\big)^{-s_c}\vec{U}_\infty^j = \begin{cases} \Re\langle\nabla\rangle^{-s_c}\vec{U}_\infty^j \text{ if } h_\infty^j = 1, \\ \Re|\nabla|^{-s_c}\vec{U}_\infty^j \text{ if } h_\infty^j = 0. \end{cases} \tag{3.11}$$

We remark that the unique existence of a local solution $\vec{U}_\infty^j$ around $t = \tau_\infty^j$ is known in all cases, including $h_\infty^j = 0$ and $\tau_\infty^j = \pm\infty$. We say that $\vec{U}_\infty^j$ on the maximal existence interval is the nonlinear profile corresponding to the free concentrating wave $(\vec{v}_n^j; t_n^j, x_n^j, h_n^j)$.

The nonlinear concentrating wave $\vec{u}_{(n)}^j$ corresponding to $\vec{v}_n^j$ is defined by

$$\vec{u}_{(n)}^j(t,x) := T_n^j \vec{U}_\infty^j\big((t-t_n^j)/h_n^j\big). \tag{3.12}$$

When $h_\infty^j = 1$, $u_{(n)}^j$ solves (1.1). While $h_\infty^j = 0$, then it solves

$$\begin{cases} (i\partial_t + \langle\nabla\rangle)\vec{u}_{(n)}^j = \big(\langle\nabla\rangle - |\nabla|\big)\vec{u}_{(n)}^j - |\nabla|^{s_c-1} f\big(|\nabla|^{-s_c}\langle\nabla\rangle^{s_c} u_{(n)}^j\big), \\ \vec{u}_{(n)}^j(0) = T_n^j \vec{U}_\infty^j(\tau_n^j). \end{cases} \tag{3.13}$$

The existence time interval of $u_{(n)}^j$ may be finite and even go to 0, however, we have

$$\begin{aligned} \|\vec{u}_n^j(0) - \vec{u}_{(n)}^j(0)\|_{L_x^2} &= \big\|T_n^j\vec{V}_n^j(\tau_n^j) - T_n^j\vec{U}_\infty^j(\tau_n^j)\big\|_{L_x^2} \\ &\leqslant \big\|\vec{V}_n^j(\tau_n^j) - \vec{V}_\infty^j(\tau_n^j)\big\|_{L_x^2} + \big\|\vec{V}_\infty^j(\tau_n^j) - \vec{U}_\infty^j(\tau_n^j)\big\|_{L_x^2} \to 0, \end{aligned} \tag{3.14}$$

as $n \to +\infty$.

Let u_n be a sequence of solutions of (1.1) around $t = 0$, and let v_n be the sequence of the free solutions with the same initial data. By Lemma 3.1, we have the linear profile decomposition for $\{\vec{v}_n\}$ as follows

$$\vec{v}_n = \sum_{j=0}^{k-1} \vec{v}_n^j + \vec{\omega}_n^k, \quad \vec{v}_n^j = e^{i\langle\nabla\rangle(t-t_n^j)} T_n^j \varphi^j.$$

Now we define the nonlinear profile decomposition as follows.

Definition 3.1 (Nonlinear profile decomposition)**.** *Let $\{\vec{v}_n^j\}_{n\in\mathbb{N}}$ be the free concentrating wave, and $\{\vec{u}_{(n)}^j\}_{n\in\mathbb{N}}$ be the sequence of the nonlinear concentrating wave corresponding to $\{\vec{v}_n^j\}_{n\in\mathbb{N}}$. Then we define the nonlinear profile decomposition of u_n by*

$$\vec{u}_{(n)}^{<k} := \sum_{j=0}^{k-1} \vec{u}_{(n)}^j = \sum_{j=0}^{k-1} T_n^j \vec{U}_\infty^j\big((t-t_n^j)/h_n^j\big). \tag{3.15}$$

We will show that $\vec{u}_{(n)}^{<k} + \vec{\omega}_n^k$ is a good approximation for $\vec{u}_n$ provided that each nonlinear profile has finite global Strichartz norm.

Next we define the Strichartz norms for the nonlinear profile decomposition. Recall that $ST(I)$ and $ST^*(I)$ are the functions spaces on $I\times\mathbb{R}^d$ defined as above

$$\begin{aligned} ST(I) =&[W](I) = L_t^{\frac{2(d+1)}{d-1}}(I; B^{s_c-\frac12}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)),\\ ST^*(I) =&[W]^*(I) \oplus L_t^1(I; B^{s_c-1}_{2,2}(\mathbb{R}^d)). \end{aligned}$$

And the Strichartz norm for the nonlinear profile $\hat{U}_\infty^j$ is defined by

$$ST_\infty^j(I) := \begin{cases} ST(I) & \text{if } h_\infty^j = 1,\\ L_t^q(I; \dot{B}^{\frac{d-3}{2}}_{q,2})\ \big(q=\frac{2(d+1)}{d-1}\big) & \text{if } h_\infty^j = 0. \end{cases} \tag{3.16}$$

The following two lemmas derive from Lemma 3.1 and the perturbation lemma. The first lemma concerns the orthogonality in the Strichartz norms.

Lemma 3.2. *Assume that in* (3.15)*, we have*

$$\|\hat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})} + \|\vec{U}_\infty^j\|_{L_t^\infty L_x^2(\mathbb{R})} < +\infty,\ \forall\ j<k. \tag{3.17}$$

Then, for any finite interval $I, j<k$, one has

$$\overline{\lim_{n\to\infty}} \|u_{(n)}^j\|_{ST(I)} \lesssim \|\hat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})}, \tag{3.18}$$

$$\overline{\lim_{n\to\infty}} \|u_{(n)}^{<k}\|_{ST(I)}^2 \lesssim \overline{\lim_{n\to\infty}} \sum_{j=0}^{k-1} \big\|u_{(n)}^j\big\|_{ST(\mathbb{R})}^2, \tag{3.19}$$

where the implicit constants is independent of I and j. Furthermore, we have

$$\lim_{n\to\infty} \Big\| f\big(u_{(n)}^{<k}\big) - \sum_{j=0}^{k-1} \Big(\frac{\langle\nabla\rangle_\infty^j}{\langle\nabla\rangle}\Big)^{s_c-1} f\Big(\Big(\frac{\langle\nabla\rangle}{\langle\nabla\rangle_\infty^j}\Big)^{s_c} u_{(n)}^j\Big)\Big\|_{ST^*(I)} = 0, \tag{3.20}$$

where $f(u)=|u|^2u$.

Proof. **Proof of** (3.18)**: Case 1:** $h_\infty^j = 1$.

It is easy to see that $u_{(n)}^j$ is just a sequence of space-time translations of $\hat{U}_\infty^j$ in this case. And so (3.18) follows in this case.

Case 2: $h_\infty^j = 0$**.**

We drop the superscript j in the following. Using the definition of $u_{(n)}$ and $\hat{U}_\infty$, we derive

$$u_{(n)}(t,x) = h_n^{s_c} T_n |\nabla|^{s_c} \langle\nabla\rangle_n^{-s_c} \widehat{U}_\infty\big((t-t_n)/h_n\big).$$

By Sobolev embedding $\dot{B}^0_{p,2}\subset L^p$ with $p\geqslant 2$ in the lower frequencies and scaling, one has for $s:=s_c-\frac12=\frac{d-3}{2},\ p=\frac{2(d+1)}{d-1}$,

$$\begin{aligned}
\big\|u_{(n)}\big\|_{B^s_{p,2}}\simeq&\big\|u_{(n)}\big\|_{L^p}+\big\|2^{js}\|\Delta_j u_{(n)}\|_{L^p}\big\|_{l^2_{j\in\mathbb{N}}}\\
\lesssim&\big\|\|\Delta_j u_{(n)}\|_{L^p}\big\|_{l^2_{j\in\mathbb{Z}^-}}+\big\|2^{js}\|\Delta_j u_{(n)}\|_{L^p}\big\|_{l^2_{j\in\mathbb{N}}}\\
\lesssim&\Big\|2^{js}\big\|\Delta_j|\nabla|^{-s}\langle\nabla\rangle^s u_{(n)}\big\|_{L^p}\Big\|_{l^2_{j\in\mathbb{Z}}}\\
\lesssim&h_n^{s_c-s-\frac d2+\frac dp}\Big\|2^{js}\big\|\Delta_j|\nabla|^{s_c-s}\langle\nabla\rangle_n^{s-s_c}\widehat{U}_\infty\big((t-t_n)/h_n\big)\big\|_{L^p}\Big\|_{l^2_{j\in\mathbb{Z}}}\\
\lesssim&h_n^{s_c-s-\frac d2+\frac dp}\Big\|2^{js}\big\|\Delta_j\widehat{U}_\infty\big((t-t_n)/h_n\big)\big\|_{L^p}\Big\|_{l^2_{j\in\mathbb{Z}}}\\
\lesssim&h_n^{s_c-s-\frac d2+\frac dp}\Big\|\widehat{U}_\infty\big((t-t_n)/h_n\big)\Big\|_{\dot{B}^s_{p,2}}.
\end{aligned}$$

Therefore, we obtain by scaling

$$\begin{aligned}
\big\|u_{(n)}\big\|_{[W](I)}\lesssim&h_n^{s_c-s-\frac d2+\frac dp}\Big\|\big\|\widehat{U}_\infty\big((t-t_n)/h_n\big)\big\|_{\dot{B}^s_{p,2}}\Big\|_{L^p_t(\mathbb{R})}\\
\lesssim&\big\|\hat{U}_\infty\big\|_{L^p_t(\mathbb{R};\dot{B}^s_{p,2})}=\big\|\hat{U}_\infty\big\|_{ST_\infty(\mathbb{R})},
\end{aligned}$$

which concludes the proof of (3.18).

Proof of (3.19)**:** We estimate the left hand side of (3.19) by

$$\begin{aligned}
\big\|u^{<k}_{(n)}\big\|^2_{ST(I)}=&\Big\|\sum_{j<k:h^j_\infty=1}u^j_{(n)}+\sum_{j<k:h^j_\infty=0}u^j_{(n)}\Big\|^2_{ST(I)}\\
\lesssim&\Big\|\sum_{j<k:h^j_\infty=1}u^j_{(n)}\Big\|^2_{ST(I)}+\Big\|\sum_{j<k:h^j_\infty=0}u^j_{(n)}\Big\|^2_{ST(I)}.
\end{aligned}$$

For the case $h^j_\infty=1$. Define $\widehat{U}^j_{\infty,R},\ u^j_{(n),R}$ and $u^{<k}_{(n),R}$ by

$$\widehat{U}^j_{\infty,R}=\chi_R\widehat{U}^j_\infty,\ u^j_{(n),R}=T^j_n\widehat{U}^j_{\infty,R},\ u^{<k}_{(n),R}:=\sum_{j<k}u^j_{(n),R},$$

where $\chi_R(t,x)=\chi(t/R,x/R)$ and $\chi(t,x)\in C^\infty_c(\mathbb{R}^{1+d})$ is the cut-off defined by

$$\chi(t,x)=\begin{cases}1, & |(t,x)|\leqslant 1,\\ 0, & |(t,x)|\geqslant 2.\end{cases}$$

Then we have

$$\Big\|\sum_{j<k:h^j_\infty=1}u^j_{(n)}\Big\|^2_{ST(I)}\lesssim\Big\|\sum_{j<k:h^j_\infty=1}u^j_{(n),R}\Big\|^2_{ST(I)}+\Big\|\sum_{j<k:h^j_\infty=1}u^j_{(n)}-\sum_{j<k:h^j_\infty=1}u^j_{(n),R}\Big\|^2_{ST(I)}.$$

On one hand, we know that

$$\Big\|\sum_{j<k:h^j_\infty=1}u^j_{(n)}-\sum_{j<k:h^j_\infty=1}u^j_{(n),R}\Big\|_{ST(I)}\leqslant\sum_{j<k:h^j_\infty=1}\Big\|(1-\chi_R)\widehat{U}^j_\infty\Big\|_{ST(\mathbb{R})}\to\ 0,$$

as $R \to +\infty$. On the other hand, by (3.6), the similar orthogonality and approximation analysis as in [14], we obtain

$$\varlimsup_{n\to\infty}\Big\|\sum_{j<k:h_\infty^j=1} u^j_{(n)}\Big\|^2_{ST(I)} \lesssim \varlimsup_{n\to\infty}\Big\|\sum_{j<k:h_\infty^j=1} u^j_{(n)}\Big\|^2_{ST(I)} \lesssim \varlimsup_{n\to\infty}\sum_{j<k:h_\infty^j=1}\Big\|u^j_{(n)}\Big\|^2_{ST(I)},$$

and for the case $h^j_\infty = 0$

$$\varlimsup_{n\to\infty}\Big\|\sum_{j<k:h_\infty^j=0} u^j_{(n)}\Big\|^2_{ST(I)} \lesssim \varlimsup_{n\to\infty}\sum_{j<k:h_\infty^j=0}\Big\|u^j_{(n)}\Big\|^2_{ST(I)}.$$

Proof of (3.20)**:** By the definition of $u^j_{(n)}$ and $\hat{U}^j_\infty$, we know that

$$u^j_{(n)}(x,t) = \Re\langle\nabla\rangle^{-s_c}\vec{u}^j_{(n)}(t,x) = \Re\langle\nabla\rangle^{-s_c}T^j_n\vec{U}^j_\infty\Big(\frac{t-t^j_n}{h^j_n}\Big) = (h^j_n)^{s_c}T^j_n\Big(\frac{\langle\nabla\rangle^j_\infty}{\langle\nabla\rangle^j_n}\Big)^{s_c}\hat{U}^j_\infty\Big(\frac{t-t^j_n}{h^j_n}\Big).$$

Let $u^{<k}_{\langle n\rangle}(t,x) = \sum\limits_{j<k} u^j_{\langle n\rangle}(x,t)$, where $u^j_{\langle n\rangle}(x,t)$ is defined by

$$u^j_{\langle n\rangle}(x,t) = \Big(\frac{\langle\nabla\rangle}{\langle\nabla\rangle^j_\infty}\Big)^{s_c}u^j_{(n)} = (h^j_n)^{s_c}T^j_n\hat{U}^j_\infty\Big(\frac{t-t^j_n}{h^j_n}\Big).$$

Then we have

$$\begin{aligned}
&\Big\|f\big(u^{<k}_{(n)}\big) - \sum_{j=0}^{k-1}\Big(\frac{\langle\nabla\rangle^j_\infty}{\langle\nabla\rangle}\Big)^{s_c-1}f\Big(\Big(\frac{\langle\nabla\rangle}{\langle\nabla\rangle^j_\infty}\Big)^{s_c}u^j_{(n)}\Big)\Big\|_{ST^*(I)}\\
\leqslant&\big\|f\big(u^{<k}_{(n)}\big) - f\big(u^{<k}_{\langle n\rangle}\big)\big\|_{ST^*(I)} + \big\|f(u^{<k}_{\langle n\rangle}) - \sum_{j<k}f(u^j_{\langle n\rangle})\big\|_{ST^*(I)}\\
&+\Big\|\sum_{j<k}f(u^j_{\langle n\rangle}) - \sum_{j<k}\Big(\frac{\langle\nabla\rangle^j_\infty}{\langle\nabla\rangle}\Big)^{s_c-1}f(u^j_{\langle n\rangle})\Big\|_{ST^*(I)}\\
\leqslant&\big\|f\big(u^{<k}_{(n)}\big) - f\big(u^{<k}_{\langle n\rangle}\big)\big\|_{ST^*(I)} + \big\|f(u^{<k}_{\langle n\rangle}) - \sum_{j<k}f(u^j_{\langle n\rangle})\big\|_{ST^*(I)} &(3.21)\\
&+\Big\|\sum_{j<k:h^j_\infty=0}f(u^j_{\langle n\rangle}) - \sum_{j<k:h^j_\infty=0}\Big(\frac{|\nabla|}{\langle\nabla\rangle}\Big)^{s_c-1}f(u^j_{\langle n\rangle})\Big\|_{ST^*(I)}. &(3.22)
\end{aligned}$$

Using (3.6) and the approximation argument in [14], we get

$$(3.21) \to 0$$

as $n \to \infty$. In addition, by $h^j_n \to 0$ as $n \to \infty$, one has

$$\begin{aligned}
&\Big\|\sum_{j<k:h^j_\infty=0}\Big(1-\Big(\frac{|\nabla|}{\langle\nabla\rangle}\Big)^{s_c-1}\Big)f\big(u^j_{\langle n\rangle}\big)\Big\|_{ST^*(I)}\\
\lesssim&\sum_{j<k:h^j_\infty=0}\Big\|\Big(1-\Big(\frac{|\nabla|}{\langle\nabla\rangle^j_n}\Big)^{s_c-1}\Big)f\big(\hat{U}^j_\infty\big)\Big\|_{ST^*(I)} \to 0
\end{aligned}$$

as $n \to \infty$. Hence we obtain (3.20). And so we complete the proof of this lemma. □

With this preliminaries in hand, we now show that $\vec{u}^{<k}_{(n)}+\vec{\omega}^k_n$ is a good approximation for $\vec{u}_n$ provided that each nonlinear profile has finite global Strichartz norm.

Lemma 3.3. *Assume that u_n is a sequence of local solutions of* (1.1) *around $t = 0$ obeying $\varlimsup\limits_{n\to\infty}\|(u_n,\dot u_n)\|_{L^\infty_t(I_n;H^{s_c}_x\times H^{s_c-1}_x)} < +\infty$. Assume also that in its nonlinear profile decomposition* (3.15)*, every nonlinear profile $\vec{U}^j_\infty$ has finite global Strichartz and L^2_x norms; that is*

$$\|\hat{U}^j_\infty\|_{ST^j_\infty(\mathbb{R})} + \|\vec{U}^j_\infty\|_{L^\infty_t L^2_x(\mathbb{R})} < +\infty. \tag{3.23}$$

Then u_n is bounded for large n in the Strichartz and the H^{s_c} norms, i.e.

$$\varlimsup_{n\to\infty}\big(\|u_n\|_{ST(\mathbb{R})} + \|\vec{u}_n\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^d)}\big) < +\infty. \tag{3.24}$$

Proof. We only need to verify the conditions of Lemma 2.5. For this purpose, by (3.13), we derive that $u^{<k}_{(n)}+\omega^k_n$ satisfies that

$$(i\partial_t+\langle\nabla\rangle)\big(\vec{u}^{<k}_{(n)}+\vec{\omega}^k_n\big) = -\langle\nabla\rangle^{s_c-1}\Big[f(u^{<k}_{(n)}+\omega^k_n) + eq\big(u^{<k}_{(n)},\omega^k_n\big)\Big],$$

where the error term $eq\big(u^{<k}_{(n)},\omega^k_n\big)$ is

$$\begin{aligned} eq\big(u^{<k}_{(n)},\omega^k_n\big) =& \sum_{j<k}\langle\nabla\rangle^{1-s_c}\big(\langle\nabla\rangle-\langle\nabla\rangle^j_\infty\big)\vec{u}^j_{(n)} + \Big[f(u^{<k}_{(n)}+\omega^k_n) - f\big(u^k_{(n)}\big)\Big] \\ &+ f(u^{<k}_{(n)}) - \sum_{j=0}^{k-1}\Big(\frac{\langle\nabla\rangle^j_\infty}{\langle\nabla\rangle}\Big)^{s_c-1} f\Big(\Big(\frac{\langle\nabla\rangle}{\langle\nabla\rangle^j_\infty}\Big)^{s_c}u^j_{(n)}\Big). \end{aligned}$$

First, by the definition of the nonlinear concentrating wave $u^j_{(n)}$ and (3.14), we have

$$\Big\|\big(\vec{u}^{<k}_{(n)}(0)+\vec{w}^k_n(0)\big) - \vec{u}_n(0)\Big\|_{L^2_x} \leqslant \sum_{j=0}^{k-1}\big\|\vec{u}^j_{(n)}(0)-\vec{u}^j_n(0)\big\|_{L^2_x}\to 0,$$

as $n\to+\infty$. This verifies the condition (2.17) by the Strichartz estimate.

Next, by the linear profile decomposition in Lemma 3.1, we get by (3.7)

$$\|\vec{u}_n(0)\|^2_{L^2} = \|\vec{v}_n(0)\|^2_{L^2} \geqslant \sum_{j=0}^{k-1}\|\vec{v}^j_n(0)\|^2_{L^2} + o_n(1) = \sum_{j=0}^{k-1}\|\vec{u}^j_{(n)}(0)\|^2_{L^2} + o_n(1). \tag{3.25}$$

Hence except for a finite set $J\subset\mathbb{N}$, the $H^{s_c}_x\times H^{s_c-1}_x$-norm of $\big(u^j_{(n)}(0),\dot u^j_{(n)}(0)\big)$ with $j\notin J$ is smaller than the iteration threshold (the small data scattering, Theorem 2.1), and so

$$\|u^j_{(n)}\|_{ST(\mathbb{R})}\lesssim\|\vec{u}^j_{(n)}(0)\|_{L^2_x},\quad j\notin J.$$

This together with (3.18), (3.19), (3.23) and (3.25) yield that for any finite interval I

$$\begin{aligned} \sup_k\varlimsup_{n\to\infty}\|u^{<k}_{(n)}\|^2_{ST(I)} &\lesssim \sum_{j\in J}\|u^j_{(n)}\|^2_{ST(I)} + \sum_{j\notin J}\|u^j_{(n)}\|^2_{ST(\mathbb{R})} \\ &\lesssim \sum_{j\in J}\|\hat{U}^j_\infty\|^2_{ST^j_\infty(\mathbb{R})} + \varlimsup_{n\to\infty}\|\vec{u}_n(0)\|^2_{L^2} < +\infty. \end{aligned} \tag{3.26}$$

This together with the Strichartz estimate for ω_n^k implies that

$$\sup_k \varlimsup_{n\to\infty} \|u_{(n)}^{<k} + \omega_n^k\|_{ST(I)} < +\infty.$$

By the similar argument as above, we obtain for large n

$$\big\|\vec{u}_{(n)}^{<k} + \vec{w}_n^k\big\|_{L_t^\infty L_x^2} \leqslant E_0.$$

Hence we verify the conditions (2.15) and (2.16).

It remains to verify the condition (2.18). Using Lemma 3.1 and Lemma 3.2, we have

$$\big\|f(u_{(n)}^{<k} + \omega_n^k) - f(u_{(n)}^{<k})\big\|_{ST^*(I)} \to 0,$$

and

$$\bigg\|f\big(u_{(n)}^{<k}\big) - \sum_{j=0}^{k-1}\Big(\frac{\langle\nabla\rangle_\infty^j}{\langle\nabla\rangle}\Big)^{s_c-1} f\Big(\Big(\frac{\langle\nabla\rangle}{\langle\nabla\rangle_\infty^j}\Big)^{s_c} u_{(n)}^j\Big)\bigg\|_{ST^*(I)} \to 0,$$

as $n \to +\infty$. On the other hand, the linear part in $eq\big(u_{(n)}^{<k}, \omega_n^k\big)$ vanishes if $h_\infty^j = 1$, and from (3.8), we know that it is controlled if $h_\infty^j = 0$ by

$$\begin{aligned}
\Big\|\langle\nabla\rangle^{1-s_c}\big(\langle\nabla\rangle - |\nabla|\big)\vec{u}_{(n)}^j\Big\|_{L_t^1(I;B_{2,2}^{s_c-1})} \lesssim & |I|\cdot\Big\|\langle\nabla\rangle^{-1}\vec{u}_{(n)}^j\Big\|_{L_t^\infty(\mathbb{R};L_x^2)} \\
\simeq & |I|\cdot\Big\|\langle\nabla/h_n^j\rangle^{-1}\vec{U}_\infty^j\Big\|_{L_t^\infty(\mathbb{R};L_x^2)} \\
\to & 0, \quad \text{as} \quad n\to+\infty,
\end{aligned} \tag{3.27}$$

by the continuity in t and Lebesgue domainted convergence theorem for bounded t, and by the scattering of $\hat{U}_\infty^j$ for $t \to \pm\infty$, which follows from $\big\|\hat{U}_\infty^j\big\|_{ST_\infty^j(\mathbb{R})} < +\infty$, and again Lebesgue domainted convergence theorem.

Thus, $\big\|eq\big(u_{(n)}^{<k}, \omega_n^k\big)\big\|_{ST^*(I)} \to 0$, as $n \to +\infty$.

Therefore, for k sufficiently close to K and n large enough, the true solution u_n and the near solution $u_{(n)}^{<k} + \omega_n^k$ satisfy all the assumptions of the perturbation Lemma 2.5. Thus, we conclude this Theorem. □

4. Concentration Compactness

Using the profile decomposition in the previous section and the perturbation theory, we argue in this section that if the scattering result does not hold, then there must exist a minimal solution with some good compactness properties.

Proposition 4.1. *Suppose that $E_c < +\infty$. Then there exists a global solution u_c of* (1.1) *satisfying*

$$\sup_{t\in\mathbb{R}} \big\|(u_c, \dot{u}_c)\big\|_{H^{s_c}\times H^{s_c-1}} = E_c, \text{ and } \|u_c\|_{ST(\mathbb{R})} = +\infty. \tag{4.1}$$

Moreover, there exists $x(t) : \mathbb{R} \to \mathbb{R}^d$ such that the set $K = \big\{(u_c, \dot{u}_c)(t, x - x(t)) \;\big|\; t \in \mathbb{R}^+\big\}$ is precompact in $H^{s_c}(\mathbb{R}^d) \times H^{s_c-1}(\mathbb{R}^d)$.

Proof. By the definition of E_c, we can choose a sequence of solutions to (1.1): $\{u_n(t):\ I_n\times\mathbb{R}\to\mathbb{R}\}$ such that

$$\sup_{t\in I_n}\big\|(u_n,\dot{u}_n)\big\|_{H^{s_c}\times H^{s_c-1}}\to E_c,\ \text{and}\ \|u_n\|_{ST(I_n)}\to+\infty,\ \text{as}\ n\to+\infty. \tag{4.2}$$

By Lemma 3.1, we have

$$\begin{cases} e^{it\langle\nabla\rangle}\vec{u}_n(0)=\sum\limits_{j=0}^{k-1}\vec{v}_n^j+\vec{w}_n^k,\ \vec{v}_n^j=e^{i\langle\nabla\rangle(t-t_n^j)}T_n^j\varphi^j(x),\\ u_{(n)}^{<k}=\sum\limits_{j=0}^{k-1}u_{(n)}^j,\ \vec{u}_{(n)}^j(t,x)=T_n^j\vec{U}_\infty^j\big((t-t_n^j)/h_n^j\big),\\ \|\vec{v}_n^j(0)-\vec{u}_{(n)}^j(0)\|_{L_x^2}\to0,\ as\ n\to+\infty. \end{cases} \tag{4.3}$$

Observing that

(1) it follows from the definition of E_c that every solution of (1.1) with $L_t^\infty(I;H_x^{s_c}\times H_x^{s_c-1})$-norm less than E_c has global finite Strichartz norm.
(2) Lemma 3.3 precludes that all the nonlinear profiles $\vec{U}_\infty^j$ have finite global Strichartz norm.

and by (3.7), we derive that there is only one profile, i.e. $K=1$, and so for large n

$$\sup_{t\in I}\big\|(u_{(n)}^0,\dot{u}_{(n)}^0)\big\|_{H^{s_c}\times H^{s_c-1}}=E_c,\ \|\hat{U}_\infty^0\|_{ST_\infty^0(I)}=+\infty,\ \lim_{n\to+\infty}\|\vec{\omega}_n^1\|_{L_t^\infty L_x^2}=0. \tag{4.4}$$

If $h_n^0\to0$, then $\hat{U}_\infty^0=\Re|\nabla|^{-s_c}\vec{U}_\infty^0$ solves the $\dot{H}_x^{s_c}(\mathbb{R}^d)$-critical wave equation

$$\partial_{tt}u-\Delta u+|u|^2u=0$$

and satisfies

$$\sup_{t\in I}\big\|(\hat{U}_\infty^0,\partial_t\hat{U}_\infty^0)\big\|_{H^{s_c}\times H^{s_c-1}}=E_c<+\infty,\ \big\|\hat{U}_\infty^0\big\|_{L_t^q(I;\dot{B}_{q,2}^{\frac{d-3}{2}})}=+\infty,\ q=\frac{2(d+1)}{d-1}.$$

But Bulut has shown that there is no such solution in [5,6]. Therefore, $h_n^0\equiv1$. And so there exist a sequence $(t_n,x_n)\in\mathbb{R}\times\mathbb{R}^d$ and $\phi\in L^2(\mathbb{R}^d)$ such that along some subsequence,

$$\big\|\vec{u}_n(0,x)-e^{-it_n\langle\nabla\rangle}\phi(x-x_n)\big\|_{L_x^2}\to0,\ \text{as}\ n\to+\infty. \tag{4.5}$$

Now we show that $\hat{U}_\infty^0=\Re\langle\nabla\rangle^{-s_c}\vec{U}_\infty^j$ is a global solution. If not, then there exist a sequence $t_n\in\mathbb{R}$ which approaches the maximal existence time. Noting that $\big(\hat{U}_\infty^0(t+t_n),\partial_t\hat{U}_\infty^0(t+t_n)\big)$ satisfies (4.2), and then by the same argument as (4.5), we deduce that there exist another sequence $(t_n',x_n')\in\mathbb{R}\times\mathbb{R}^d$ and for some $\psi\in L^2$ so that

$$\big\|\vec{U}_\infty^0(t_n)-e^{-it_n'\langle\nabla\rangle}\psi(x-x_n')\big\|_{L_x^2}\to0, \tag{4.6}$$

as $n\to\infty$. We write $\vec{v}:=e^{it\langle\nabla\rangle}\psi$. From Strichartz estimate, we know that for any $\varepsilon>0$, there exist $\delta>0$ with $I=[-\delta,\delta]$ so that

$$\big\|\langle\nabla\rangle^{-s_c}\vec{v}(t-t_n')\big\|_{ST(I)}\leqslant\frac{\eta_0}{2},$$

where $\eta_0 = \eta(d)$ is the threshold from the small data theory. This together with (4.6) shows that for sufficiently large n

$$\big\| \langle \nabla \rangle^{-s_c} e^{it\langle \nabla \rangle} \vec{U}^0_\infty (t_n) \big\|_{ST(I)} \leqslant 2\eta_0.$$

Hence, by the small data theory (Theorem 2.1), we derive that the solution $\vec{U}^0_\infty$ exists on $[t_n - \delta, t_n + \delta]$ for large n, which contradicts with the choice of t_n. Thus $\hat{U}^0_\infty$ is a global solution and it is just the desired critical element u_c. Moreover, since (1.1) is symmetric in t, we may assume that

$$\|u_c\|_{ST(0,+\infty)} = +\infty. \tag{4.7}$$

We call such u a forward critical element.

Next we prove the precompactness of K. It is equivalent to show the precompactness of $\{\vec{u}(t_n)\}$ in L^2_x for any $t_1, t_2, \cdots > 0$. It is easy to prove this by the continuity in t when t_n converges. Thus, we can suppose that $t_n \to +\infty$. Applying the property of (4.5) to the sequence of solution $\vec{u}(t + t_n)$, we get another sequence $(t'_n, x'_n) \in \mathbb{R}^d$ and $\phi \in L^2$ such that

$$\big\| \vec{u}(t_n, x) - e^{-it'_n \langle \nabla \rangle} \phi(x - x'_n) \big\|_{L^2_x} \to 0, \ n \to \infty. \tag{4.8}$$

If $t'_n \to -\infty$, then we obtain by triangle inequality

$$\begin{aligned} \|\langle \nabla \rangle^{-s_c} e^{it\langle \nabla \rangle} \vec{u}(t_n)\|_{ST(0,\infty)} \leqslant & \|\langle \nabla \rangle^{-s_c} e^{it\langle \nabla \rangle} \big(\vec{u}(t_n) - e^{-it'_n\langle \nabla \rangle} \phi(x - x'_n) \big) \|_{ST(0,\infty)} \\ & + \|\langle \nabla \rangle^{-s_c} e^{i(t - t'_n)\langle \nabla \rangle} \phi(x - x'_n) \|_{ST(0,\infty)} \\ \lesssim & \|\vec{u}(t_n, x) - e^{-it'_n\langle \nabla \rangle} \phi(x - x'_n)\|_{L^2_x} + \|\langle \nabla \rangle^{-s_c} e^{it\langle \nabla \rangle} \phi\|_{ST(-t'_n,\infty)} \\ \to & \, 0. \end{aligned}$$

Thus, by the small data theory, we can solve u for $t < t_n$ with large n globally, which contradicts with its forward criticality.

If $t'_n \to +\infty$, then one has

$$\big\| \langle \nabla \rangle^{-s_c} e^{it\langle \nabla \rangle} \vec{u}(t_n) \big\|_{ST(-\infty,0)} = \big\| \langle \nabla \rangle^{-s_c} e^{it\langle \nabla \rangle} \phi \big\|_{ST(-\infty,-t'_n)} + o(1) \to 0.$$

Hence, we can solve u for $t < t_n$ with large n with diminishing Strichartz norms. We give a contradiction since $u = 0$ by taking the limit.

Thus, t'_n is bounded, which shows that $\{t'_n\}$ is precompact, so is $\vec{u}(t_n, x + x'_n)$ in L^2_x by (4.8). $\square$

As a direct consequence of the above proposition, we have

Corollary 4.1. *(Compactness) Let u be a forward critical element. Then, for any $\eta > 0$, there exist $x(t) : \mathbb{R}^+ \to \mathbb{R}^d$ and $C(\eta) > 0$ such that*

$$\sup_{t \in \mathbb{R}^+} \int_{|x - x(t)| \geqslant C(\eta)} \Big(\big| \langle \nabla \rangle^{s_c} u \big|^2 + \big| \langle \nabla \rangle^{s_c - 1} \dot{u} \big|^2 \Big) dx \leqslant \eta. \tag{4.9}$$

We refer to the function $x(t)$ as the spatial center function, and to $C(\eta)$ as the compactness modules function.

We remark that the small data theory shows that the $H^{s_c}_x(\mathbb{R}^d) \times H^{s_c-1}_x(\mathbb{R}^d)$ norm of a blowup solution must remain bounded from below. The fact that this norm is nonlocal

in odd space dimensions reduces the efficacy of this statement. Our next lemma gives a lower bound in a more suitable norm, and also a mild control of $x(t)$.

Lemma 4.1. *Let u be a nonlinear strong solution of* (1.1) *as in Proposition 4.1. Then*

(1) (($\nabla_{t,x}u, u$) *nontrivially*) *We have*

$$\inf_{t\in\mathbb{R}^+}\int_{\mathbb{R}^d}\Big(|u|^{\frac{d}{2}}+|\nabla_{t,x}u|^{\frac{d}{2}}\Big)dx\gtrsim 1. \tag{4.10}$$

(2) (*Control of $x(t)$*) *For some large constant C_u, we have for any $t_1,t_2\in\mathbb{R}^+$*

$$|x(t_1)-x(t_2)|\leqslant|t_1-t_2|+2C_u. \tag{4.11}$$

Proof. The proof is similar to [21]. But we give a sketch for the sake of completeness.

(1) It follows from the small data theory that

$$\inf_{t\in\mathbb{R}^+}\big\|(u,u_t)\big\|_{H^{s_c}\times H^{s_c-1}}\gtrsim 1, \tag{4.12}$$

otherwise u would have finite spacetime norm which contradicts with (4.1).

On the other hand, it is easy to see that

$$\frac{\|f\|_{L_x^{\frac{d}{2}}(\mathbb{R}^d)}}{\|f\|_{H_x^{s_c-1}(\mathbb{R}^d)}}>0,$$

for any nonzero $\mathbb{R}^{1+d}$-valued $f\in H_x^{s_c-1}(\mathbb{R}^d)$. We note that

$$\begin{cases}(i)\ \text{This ratio achieves a nonzero minimum on any compact set}\\ \qquad\text{that does not contain the zero function;}\\ (ii)\ \text{this ratio is invariant under translation;}\\ (iii)\ f:=\big(u_t,\langle\nabla\rangle u\big)\ \text{and the set } K \text{ is precompact in } H_x^{s_c}(\mathbb{R}^d)\times H_x^{s_c-1}(\mathbb{R}^d).\end{cases}$$

Combining these facts with (4.12), we obtain that this ratio is bounded from below, and so (4.10) follows.

(2) Choose $\eta>0$ to be a small constant below the $H_x^{s_c}(\mathbb{R}^d)\times H_x^{s_c-1}(\mathbb{R}^d)$ threshold for the small data theory. By Corollary 4.1, there is a constant $C(\eta)>0$ such that

$$\Big\|\phi\Big(\frac{x-x(t_1)}{C(\eta)}\Big)u(t_1,x)\Big\|_{H_x^{s_c}(\mathbb{R}^d)}+\Big\|\phi\Big(\frac{x-x(t_1)}{C(\eta)}\Big)u_t(t_1,x)\Big\|_{H_x^{s_c-1}(\mathbb{R}^d)}\leqslant\eta, \tag{4.13}$$

where $\phi:\mathbb{R}^d\to[0,+\infty)$,

$$\phi(x)=\begin{cases}1, & |x|\geqslant 1,\\ 0, & |x|\leqslant\frac{1}{2}.\end{cases} \tag{4.14}$$

Hence, by the small data theory, there is a global solution to (1.1) whose Cauchy data at time t_1 match the combination of ϕ and u given in (4.13). Moreover, from the small data theory, each critical Strichartz norm of this solution is controlled by a multiple of η. It follows from domain of dependence arguments that this new solution agrees with the original u on the set

$$\Omega(t):=\big\{x:\ |x-x(t_1)|\geqslant|t-t_1|+C(\eta)\big\},\quad t\in\mathbb{R}$$

and so by Sobolev embedding,

$$\big\|(u, \nabla_{t,x} u)\big\|_{L_x^{\frac{d}{2}}(\Omega(t))} \lesssim \eta, \ \forall\, t \in \mathbb{R}.$$

In particular, taking $t = t_2$, we get

$$\int_{|x-x(t_1)|\geqslant|t_2-t_1|+C(\eta)} \Big(|u(t_2,x)|^{\frac{d}{2}} + |\nabla_{t,x} u(t_2,x)|^{\frac{d}{2}}\Big)dx \lesssim \eta. \tag{4.15}$$

On the other hand, we have by Corollary 4.1 and Sobolev embedding,

$$\int_{|x-x(t_2)|\geqslant C(\eta)} \Big(|u(t_2,x)|^{\frac{d}{2}} + |\nabla_{t,x} u(t_2,x)|^{\frac{d}{2}}\Big)dx \lesssim \eta, \tag{4.16}$$

This together with (4.10) and (4.15) yield that

$$\big\{x:\ |x-x(t_1)| \leqslant |t_2-t_1| + C(\eta)\big\} \cap \big\{x:\ |x-x(t_2)| \leqslant C(\eta)\big\} \neq \emptyset.$$

This concludes the proof of (4.11). □

The next corollary shows that the potential energy of the critical element must concentrate.

Corollary 4.2 (Concentration of potential energy)**.** *Let u be a nonlinear strong solution of (1.1) such that the set K defined in Proposition 4.1 is precompact in $H^{s_c}(\mathbb{R}^d) \times H^{s_c-1}(\mathbb{R}^d)$, and $E(u,\dot u) \neq 0$. For every $\tau > 0$, there exists two positive numbers $\alpha(\tau,u)$ and $\beta(\tau,u)$ such that, for all time t, there holds that*

$$\alpha \leqslant \int_t^{t+\tau}\int_{\mathbb{R}^d} |u(s,x)|^4 dxds \leqslant \beta, \tag{4.17}$$

Moreover, combining this with Corollary 4.1 and Sobolev embedding, we have for large $C = C(u)$ and all t

$$\int_t^{t+1}\int_{|x-x(t)|\leqslant C} |u(s,x)|^4 dxds \gtrsim 1. \tag{4.18}$$

Proof. The bound from above follows from Sobolev's inequality and $\sup\limits_{t\in\mathbb{R}} \big\|(u_c, \dot u_c)\big\|_{H^{s_c}\times H^{s_c-1}} = E_c < +\infty$. Suppose the bound from below is not true. Then there exist $\tau > 0$ and a sequence t_k such that

$$\int_0^{\tau}\int_{\mathbb{R}^d} \big|u(t+t_k, x - x(t_k))\big|^4 dxdt = \int_{t_k}^{t_k+\tau}\int_{\mathbb{R}^d} |u(t,x)|^4 dxdt < \frac{1}{k}. \tag{4.19}$$

Using the precompactness of K, we can extract a subsequence and assume that

$$\big(\tau_{x(t_k)} u(t_k), \tau_{x(t_k)} u_t(t_k)\big) \ \to\ (U_0, U_1) \quad \text{in } H_x^{s_c}(\mathbb{R}^d) \times H_x^{s_c-1}(\mathbb{R}^d).$$

Let $U:\ I \times \mathbb{R}^d \to \mathbb{R}$ be the nonlinear strong solution of (1.1) with initial data (U_0, U_1) at time $t = 0$. Then, $E(U,\dot U) = E(u,\dot u) \neq 0$. By wellposedness and (4.19), we get

$$\int_{[0,\tau]\cap I}\int_{\mathbb{R}^d} |U(t,x)|^4 dxdt = 0$$

Hence, we have $U(t) = 0$ for all t in $(0,\tau) \cap I$, hence $U_t(t) = 0$ for all such t. Consequently, $E(u,\dot u) = 0$. This is a contradiction. □

5. Extinction of the critical element

In this section, we prove that the critical solution constructed in Section 4 does not exist, thus ensuring that $E_c = +\infty$. This implies Theorem 1.1.

Proposition 5.1. *There are no solutions to* (1.1) *in the sense of Proposition 4.1.*

Proof. We argue by contradiction. Assume there exists a solution $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ such that the set K defined in Proposition 4.1 is precompact in $H^{s_c}(\mathbb{R}^d)\times H^{s_c-1}(\mathbb{R}^d)$. We will show that this scenario is inconsistent with the following Morawetz inequality [4,23,25]

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|}dxdt \lesssim E(u,u_t). \tag{5.1}$$

On one hand, since the solution u has finite energy, the right-hand side in the Morawetz inequality is finite and so

$$\int_0^T\int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|}dxdt \lesssim E(u,u_t) \lesssim 1, \tag{5.2}$$

for any $T > 0$. On the other hand, we have concentration of potential energy by Corollary 4.2. That is, there exists $C = C(u)$ such that

$$\int_{t_0}^{t_0+1}\int_{|x-x(t)|\leqslant C} |u(t,x)|^4 dxdt \gtrsim 1,$$

for any $t_0 \in \mathbb{R}$. Translating space so that $x(0) = 0$ and employing finite speed of propagation in the sense (4.11)

$$|x(t) - x(0)| \leqslant |t| + 2c_u,$$

we deduce that for $T \geqslant 1$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|}dxdt &\gtrsim \int_0^T\int_{|x-x(t)|\leqslant C} \frac{|u(t,x)|^4}{|x|}dxdt\\ &\gtrsim \int_0^T \frac{dt}{1+t}\\ &\gtrsim \log(1+T),\end{aligned}$$

which contradicts with (5.2) by choosing T sufficiently large depending on u. Hence we complete the proof of this proposition. $\square$

Acknowledgements The authors were supported by the NSF of China under grant No.11171033, 11231006.

References

[1] J. Bergh and J. Löfström, Interpolation spaces, An introduciton. Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin-New York, 1976.

[2] M. S. Birman and S. Z. Solomjak, On estimates on singular number of integral operators III, Vest. LSU Math. 2 (1975), 9-27.

[3] P. Brenner, On space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations, Math. Z. 186 (1984), 383-391.

[4] P. Brenner, On scattering and everywhere defined scattering operators for nonlinear Klein-Gordon equtaons, J. Differential Equations 56 (1985), 310-344.

[5] A. Bulut, Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation. J. Func. Anal. 263 (2012), 1609-1660. MR2948225.

[6] A. Bulut, The radial defocusing energy-supercritical cubic nonlinear wave equation in $\mathbb{R}^{1+5}$. Preprint, arXiv: 1104.2002.

[7] A. Bulut, The defocusing energy-supercritical cubic nonlinear wave equation in dimension five. Preprint arXiv:1112.0629.

[8] M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100 (1991), 87-109. MR1124294.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and scattering for the energy-cirtical nonlinear Schrödinger equation in $\mathbb{R}^3$. Annals of Math. 167 (2008), 767-865.

[10] T. Duyckaerts, C.Kenig and F.Merle, Scattering for radial, bounded solutions of focusing supercritical wave equations. arXiv: 1208.2158v1.

[11] T. Duyckaerts, C.Kenig and F.Merle, Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation. J. Eur. Math. Soc., 13 (2011), 533-599.

[12] J. Ginibre and G. Velo, Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 43 (1985), 399-442.

[13] J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equation, J. Funct. Anal., 133 (1995), 50-68.

[14] S. Ibrahim, N. Masmoudi and K. Nakanishi, Scattering threshold for the focusing nonlinear Klein-Gordon equation, Analysis and PDE., 4 (2011), 405-460.

[15] S. Ibrahim, N. Masmoudi and K. Nakanishi, Threshold solutions in the case of mass-shift for the critical Klein-Gordon equation, To appear in Tran. Amer. Math. Soc., arXiv:1110.1709v1.

[16] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), 955-980.

[17] C. Kenig and F. Merle, Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear Schrödinger equation in the radial case, Invent. Math., 166 (2006), 645-675.

[18] C. Kenig and F. Merle, Nondispersive radial solutions to energy supercritical nonlinaer wave equations, with applications. Amer. J. Math., 133 (2011), 1029-1065.

[19] C. Kenig and F. Merle, Radial solutions to energy supercritical wave equations in odd dimensions. Disc. Cont. Dyn. Sys. A, 4 (2011) 1365-1381.

[20] R. Killip and M. Visan, Energy-supercritical NLS: critical $\dot{H}^s$-bounds imply scattering. Comm. Partial Differential Equations 35 (2010), 945-987. MR2753625.

[21] R. Killip and M. Visan, The defocusing energy-supercritical nonlinear wave equation in three space dimensions, Trans. Amer. Math. Soc., 363 (2011), 3893-3934.

[22] R. Killip and M. Visan, The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions. Proc. Amer. Math. Soc., 139 (2011), 1805-1817.

[23] C. Miao, Modern methods to the nonlinear wave equations, Lectures in Contemporary Mathematics, Vol.2, Science Press, Beijing, 2010.

[24] C. Miao, B. Zhang and D. Fang, Global well-posedness for the Klein-Gordon equations below the energy norm, Journal of Partial Differential Equations, 17 (2004), 97-121.

[25] C. Morawetz and W. A. Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, Comm. Pure Appl. Math., 25 (1972), 1-31.

[26] K. Nakanishi, Scattering theory for nonlinear Klein-Gordon equation with Sobolev critical power, Internat. Math. Res. Notices, 1 (1999), 31-60.

[27] K. Nakanishi, Remarks on the energy scattering for nonlinear Klein-Gordon and Schröinger equations. Tohoku Math. J., II. 53 (2001), 285-303.

[28] M. Taylor. Tools for PDE, Mathematical Surveys and Monographs, vol. 81,*Pseudodifferential operators, paradifferential operators, and layer potentials*, American Mathematical Society, Providence, RI, 2000. MR1766415 (2001g:35004)

Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing, China, 100088;
E-mail address: miao_changxing@iapcm.ac.cn

The Graduate School of China Academy of Engineering Physics, P. O. Box 2101, Beijing, China, 100088
E-mail address: zhengjiqiang@gmail.com